A note about stratified group actions

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Abstract

In this short note we aim at proving one property of constructible sheaves on quotient stratified topological stacks: namely, the fact that constructible sheaves do not see the difference between quotienting by a group G or by a quotient (G/H) where H is contractible. We also prove a relative version of this result.

Recall the definition of conically stratified topological space, from [Lur17, Definition A.5.5], and of the ∞-category of constructible sheaves with values in spaces

$$\mathrm{Cons}(Y,s;\mathcal{S})$$

([Lur17, Definition A.5.2]). For the purposes of this note, any stratified space for which the Exodromy theorem [Lur17, Theorem A.9.3] works will be enough. We denote the category of conically stratified spaces, locally of singular shape, and with stratifying poset satisfying the ascending chain condition, by

$$\mathrm{StrTop}_{\mathrm{con}}.$$

Recall 1. Recall from [Noc20] the notion of smooth stratified submersion and stratified homotopy equivalence between stratified spaces.

Recall also that, thanks to the Exodromy theorem, given any stratified map $f:(Y,s)\to(Z,t)$, the pullback functor

$$\mathrm{Cons}(Z,t;\mathcal{S})\to\mathrm{Cons}(Y,s;\mathcal{S})$$

admits a left adjoint, which we denote by $f^{\mathrm{c}}_{\sharp}$.

Recall now that by [Noc20], the smooth base change formula holds, i.e. when

$$\begin{array}{ccc}(X',t') & \xrightarrow{\alpha_0} & (X,t)\\ \big\downarrow{\scriptstyle\beta_1} & & \big\downarrow{\scriptstyle\beta_0}\\ (Y',s') & \xrightarrow{\alpha_1} & (Y,s)\end{array}$$

is a diagram of stratified spaces, and β_0,β_1 are smooth stratified submersions, then

$$\beta^{\mathrm{c}}_{1,\sharp}\alpha_0^*\to\alpha_1^*\beta^{\mathrm{c}}_{0,\sharp} \tag{1}$$

is an equivalence of functors $\mathrm{Cons}(X,t;\mathcal{S})\to\mathrm{Cons}(Y',s';\mathcal{S})$.

Recall finally that by [Noc20], the functor

$$\mathrm{Cons}(-;\mathcal{S}):\mathrm{StrTop}^{\mathrm{op}}_{\mathrm{con}}\to\mathcal{P}\mathrm{r}^{\mathrm{R}}$$

sends stratified homotopy equivalences to equivalences of ∞-categories.

Recall 2. Recall from [Noc20] the definition of the category $\mathrm{StrTStk}_{\mathrm{con}}$ of conically stratified topological stacks. The functor $\mathrm{Cons}(-;\mathcal{S})$ extends to $\mathrm{StrTStk}_{\mathrm{con}}^{\mathrm{op}}$ by right Kan extension.

A smooth stratified submersion (resp. a stratified homotopy equivalence) is map of stratified topological stacks which can be written as a colimit of maps between representables which are smooth stratified submersions (resp. stratified homotopy equivalences).

We are interested in a special class of stratified topological stacks. Let $(S,s),(G,w),(Y,t) \in \mathrm{StrTop}_{\mathrm{con}}$, $(G,w) \to (S,s)$ a smooth stratified submersion exhibiting G as a relative unstratified topological group over S (i.e. the stratification of G is induced by s, and each fiber is a topological group). Let $(Y,t) \to (S,s)$ be a stratified map, and suppose there is a stratified action of G on Y relative over S.

Then we define the quotient stack

$$Y/G = \mathrm{colim}\Big[\ \dots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; (G \times G \times Y, s_2) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; (G \times Y, s_1) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; (Y,s)\ \Big].$$

where s_i is the stratification on $\overbrace{G \times \cdots \times G}^{i} \times Y$ which is trivial on the group factors and s on the last factor. As usual, in the left direction are induced by the identity element of G in various ways, and maps in the right direction are induced by combinations of the action and the projections.

The stratification coming with this definition is denoted again by s.

By definition, we have that

$$\mathrm{Cons}(Y/G, s; \mathcal{S})$$

is the limit of the diagram

$$\dots \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathrm{Cons}(G \times G \times Y, s_2; \mathcal{S}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathrm{Cons}(G \times Y, s_1; \mathcal{S}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathrm{Cons}(Y, s; \mathcal{S})\ .$$

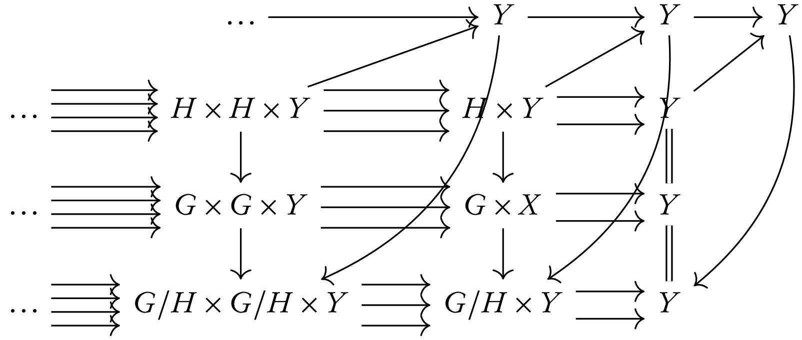
Note that smooth base change and stratified homotopy invariance follow formally from Recall 1 by taking limits.

Lemma 3. *Let G be a topological group which is also topological manifold, $H < G$ a closed normal subgroup with the same property and furthermore contractible, and (Y,s) a conically stratified topological space with a G-action such that the restriction to H is trivial. Then the pullback map*

$$\mathrm{Cons}(Y/(G/H), s) \to \mathrm{Cons}(Y/G, s),$$

where s stays for the stratification inherited by the quotients on both sides, is an equivalence.

Proof. We have a diagram



where all squares of the form

$$\begin{array}{ccc} & & X \\ & \nearrow & \\ H^n \times Y & & \\ \downarrow & & \\ G^n \times Y & & \\ \downarrow & \swarrow & \\ (G/H)^n \times Y & & \end{array}$$

are pullback squares. Note that, at each term, the map between the (H,Y) row and the $(*,Y)$ row is a stratified homotopy equivalence, and the homotopies witnessing this can be chosen to be compatible with the rest of the diagram, since the action of H on Y is trivial. Also, $G \to G/H$ is an (unstratified) Serre fibration of trivially stratified spaces (being the quotient by a free continuous action), hence also the map between the (G,Y)-row and the $(G/H,Y)$-row is a stratified homotopy equivalence. Therefore, it induces a stratified homotopy equivalence of stacks on the colimits of the two rows, and therefore an equivalence at the level of constructible sheaves by Recall 2. □

Lemma 4. *Let (S,s) be a stratified topological space, $G \to S$ a smooth stratified submersion exhibiting G as a relative topological group over S, $H \subset G$ a relative normal subgroup (with the inherited stratification) such that, for each stratum W of S, the map*

$$G \times_S W \to G/H \times_S W$$

is a stratified homotopy equivalence. Let $(Y,t) \to (S,s)$ be a stratified map of stratified topological spaces, and suppose there is a stratified action of G on Y relative over S, which restricts to the trivial action of H. Then the pullback map

$$\mathrm{Cons}(Y/(G/H), s) \to \mathrm{Cons}(Y/G, s),$$

where s stays for the stratification inherited by the quotients on both sides, is an equivalence.

Proof. As in Lemma 3, we reduce to checking that the maps

$$\pi_n^* : \mathrm{Cons}(\underbrace{(G/H) \times_S \cdots \times_S (G/H)}_{n} \times_S Y) \to \mathrm{Cons}(\underbrace{G \times_S \cdots \times_S G}_{n} \times_S Y),$$

given by pullback along the projection $\pi_n : \underbrace{G \times_S \cdots \times_S G}_{n} \times_S Y \to \underbrace{(G/H) \times_S \cdots \times_S (G/H)}_{n} \times_S Y$, are equivalences for every n. By Recall 1 there is an adjunction $\pi^{\mathrm{c}}_{n,\sharp} \dashv \pi_n^*$, hence it suffices to check that for each

$$\mathcal{F} \in \mathrm{Cons}(\underbrace{(G/H) \times_S \cdots \times_S (G/H)}_{n} \times_S Y),$$
$$\mathcal{G} \in \mathrm{Cons}(\underbrace{G \times_S \cdots \times_S G}_{n} \times_S Y)$$

the unit and counit maps

$$\mathcal{F} \to \pi_n^* \pi^{\mathrm{c}}_{n,\sharp} \mathcal{F}$$
$$\pi^{\mathrm{c}}_{n,\sharp} \pi_n^* \mathcal{G} \to \mathcal{G}$$

are equivalences. By smooth base change Eq. (1) it suffices to check this after pullback to the strata of S. There, however, the map $G \to G/H$ becomes a stratified homotopy equivalence by hypothesis, hence the claim. □

References

[Lur17] Jacob Lurie. Higher Algebra. http://people.math.harvard.edu/~lurie/papers/HA.pdf, 2017.

[Noc20] Guglielmo Nocera. A model for the $\mathbb{E}_3$ fusion-convolution product of constructible sheaves on the affine Grassmannian. https://arxiv.org/abs/2012.08504, 2020.